Phys 410
Fall 2014
Lecture #12 Summary
9 October, 2014

We considered the Lagrangian in polar coordinates for a single particle of mass m acted upon by a conservative force in two dimensions. The Lagrangian is $\mathcal{L}(r, \dot{r}, \phi, \dot{\phi}, t) = \frac{m}{2}(\dot{r}^2 + r^2\dot{\phi}^2) - U(r, \phi)$. The Euler-Lagrange equation for r yields $-\frac{\partial U}{\partial r} = m(\ddot{r} - r\dot{\phi}^2)$. This is Newton's second law for radial motion, where the first term on the right hand side is the radial acceleration, while the second is the centripetal acceleration. The Euler-Lagrange equation for φ yields $-\frac{\partial U}{\partial \varphi} = \frac{d}{dt}(mr^2\dot{\varphi})$. This is a statement that the torque acting on the particle $\left(-\frac{\partial U}{\partial \varphi} = rF_\varphi\right)$ is equal to the time rate of change of the angular momentum. In other words it is a statement of Newton's second law for rotational motion.

Constrained systems are common in physics, and their dynamics can be advantageously solved by the Lagrangian method. Examples include the pendulum, the Atwood machine, a rigid body, gas particles trapped in a box, a rolling object, and a bead on a wire. We considered the pendulum problem in detail. The constraint is that the length of the pendulum ℓ is fixed, so that the x- and y-coordinates of the bob are not independent, but constrained so that $\ell = \sqrt{x^2 + y^2}$. We can incorporate this constraint by adopting a new independent variable to describe the position of the bob, namely the angle that the pendulum makes with the vertical, ϕ. In terms of this generalized coordinate, the Lagrangian becomes $\mathcal{L}(\phi, \dot{\phi}) = \frac{m}{2}\ell^2\dot{\phi}^2 - mg\ell(1 - \cos\phi)$. Lagrange's equation gives $-mg\ell\sin\phi = m\ell^2\ddot{\phi}$, which relates the torque due to gravity on the bob to the time rate of change of the angular momentum of the bob, or the moment of inertia ($m\ell^2$) times the angular acceleration ($\ddot{\phi}$). Note that the force of constraint (namely the tension in the rod supporting the bob) never played a role in the analysis (whereas it plays an important role in the traditional Newtonian approach). Once the appropriate generalized coordinate is identified, the associated constraining force disappears from the discussion!

Generalized coordinates and constrained systems are important for Lagrangian dynamics. Consider a system consisting of N particles, with positions $\overrightarrow{r_\alpha}$, with $\alpha = 1, \ldots N$. The parameters $q_1, q_2, \ldots q_n$ are a set of generalized coordinates if each position $\overrightarrow{r_\alpha}$ can be expressed as a function of $q_1, q_2, \ldots q_n$, and possibly time t as, $\overrightarrow{r_\alpha} = \overrightarrow{r_\alpha}(q_1, q_2, \ldots q_n, t)$for $\alpha = 1, \ldots N$, and the inverse $q_i = q_i(\overrightarrow{r_1}, \overrightarrow{r_2}, \ldots \overrightarrow{r_N}, t)$ for $i = 1, 2, \ldots n$ can also be written. For particles in three dimensions, $n \le 3N$. If $n < 3N$, then the system is said to be constrained. The number of degrees of freedom of a system is the number of coordinates that can be independently varied in a small displacement. The simple pendulum is constrained and has one degree of freedom. The double pendulum is constrained and has two degrees of freedom. One can show (the proof is in Taylor, section 7.4) that constrained systems with holonomic constraints obey the Lagrange equations when their Lagrangian is written in terms of the generalized coordinates of the system. Holonomic constraints are those which impose relations between only the coordinates of the system. Non-holonomic constraints cannot be reduced to relations only between the coordinates. For example consider a rolling wheel on a fixed surface – the rolling constraint says that the velocity at the point of contact is zero. This is a condition on a quantity other than the coordinates of the particles.